# Math 246B Lecture 9 Notes

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## 1 Phragmén-Lindelöf for Strips and Cauchy's Integral Formula for Non-Holomorphic Functions

### 1.1 Phragmén-Lindelöf for a half-strip and a strip

**Proposition 1.1** (PL for a half-strip)**.** *Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0, \alpha < \operatorname{Re}(z) < \beta\}$, with $\alpha < \beta$ finite. Then $\varphi(z) = e^{k \operatorname{Im}(z)}$ is a PL function for $\Omega$ for any $0 < k < \pi/(\beta - \alpha)$.*

*Proof.* Let $F(z) = e^{-icz}$, where $c < 2\pi/(\beta - \alpha)$. $F : \Omega \to \tilde{\Omega}$ is conformal, where $\tilde{\Omega} = -\{w \in \mathbb{C} : |w| > 1, c\alpha < \arg(w) < c\beta\}$. $F$ is a homeomorphism $\overline{\Omega} \to \overline{\tilde{\Omega}}$. In $\tilde{\Omega}$, we have the PL function $\varphi(w) = |w|^{k/c}$, where $k/c < \pi/(c(\beta - \alpha))$. We get $\varphi(z) = \tilde{\varphi}(F(z)) = |F(z)|^{k/c} = e^{k \operatorname{Im}(z)}$ is a PL function for $\Omega$. $\square$

**Proposition 1.2** (PL for an entire strip)**.** *Let $\Omega = \{z \in \mathbb{C} : \alpha < \operatorname{Re}(z) < \beta\}$ with $\alpha < \beta$ finite. Then $\varphi(z) = e^{k \operatorname{Im}(z)}$ is a PL function for $\Omega$ for any $0 < k < \pi/(\beta - \alpha)$. Then $\varphi(z) = e^{k |\operatorname{Im}(z)|}$ is a PL function for $\Omega$ for any $0 < k < \pi/(\beta - \alpha)$.*

*Proof.* Let $u \in SH(\Omega)$ be upper semicontinuous on $\overline{\Omega}$, $u \leq M$ on $\partial\Omega$, and $u(z) \leq \varphi(z)$ for large $z \in \overline{\Omega}$. We want to show that $u \leq M$ on $\overline{\Omega}$. By the previous result, we get that $u \leq \max(M, L)$ on $\Omega_1 = \Omega \cap \{z : \operatorname{Im}(z) > 0\}$, where $L = \max_{[\alpha,\beta]} u < \infty$. Similarly, using $z \mapsto -z$, we conclude that $u \leq \max(M, L)$ on $\Omega_2 = \Omega \cap \{z : \operatorname{Im}(z) < 0\}$. So $u$ is bounded on $\Omega$.

We claim that any positive constant is a PL-function for $\Omega$. It suffices to construct a harmonic $\psi \geq 0$ such that $\psi(z) \to \infty$ as $|z| \to \infty$. We can take $\psi(z) = \operatorname{Re}(\sqrt{z - \gamma})$, where $\gamma < \alpha$. Then $\psi(z) = |z - \gamma|^{1/2} \cos(\arg(z - \gamma)/2) \sim |z^{1/2}|$ at $\infty$ in $\Omega$. We conclude that $u \leq M$ on $\overline{\Omega}$. So $\varphi(z) = e^{k|\operatorname{Im}(z)|}$ is a PL function for $\Omega$. $\square$

**Corollary 1.1** (Hadamard's three line theorem)**.** *Let $\Omega = \{z \in \mathbb{C} : \alpha < \operatorname{Re}(z) < \beta\}$. Let $u \in SL(\Omega)$, upper semicontinuous on $\overline{\Omega}$, $u \leq A$ on $\partial\Omega$, and $u(z) \leq e^{k|\operatorname{Im}(z)|}$ for large $z \in \Omega$, where $0 < k < \pi/(\beta - \alpha)$. Let $M(x) = \sup_{\operatorname{Re}(z) = x} u(z)$ for $\alpha \leq x \leq \beta$. Then $M$ is convex.*

The proof is similar to ideas we've seen before, so we will just give the idea.

*Proof.* Here is the idea. Let $a, b \in \mathbb{R}$ be such that $\tilde{M}(x) = M(x) - ax - b \leq 0$ for $x = \alpha, \beta$. Show that $\tilde{M}(x) \leq 0$ for $\alpha \leq x \leq \beta$. If $\tilde{u}(z) = u(z) - a\,\mathrm{Re}(z) - b$, then $\tilde{u} \in SH(\Omega)$ has the right growth at $\infty$, and $\tilde{M}(x) = \sup_{\mathrm{Re}(z)=x}(z) \implies \tilde{u} \leq 0$ on $\partial\Omega$. By the PL theorem applied to $\tilde{u}$, $\tilde{u} \leq 0$ in $\Omega$. So $\tilde{M}(x) \leq 0$ on $[\alpha, \beta]$. □

## 1.2 Cauchy's integral formula for non-holomorphic functions

**Theorem 1.1** (Cauchy's integral formula for non-holomorphic functions)**.** *Let $\omega \subseteq \mathbb{C}$ be a bounded open set with piecewise $C^1$ boundary, and let $u \in C^1(\overline{\Omega})$. Then*

$$u(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta 0 \frac{1}{\pi} \iint_{\omega} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} L(d\zeta),$$

*where $L(d\zeta)$ is the Lebesgue measure in $\omega$.*

**Remark 1.1.** The integral over $\omega$ makes sense, as $1/\zeta \in L^1_{\mathrm{loc}}(\mathbb{C})$:

$$\iint_{|\zeta|<1} \frac{1}{|\zeta|} L(d\zeta) \overset{\zeta = re^{it}}{=} \iint dr\, dt < \infty.$$

*Proof.* Let $v \in C^1(\overline{\omega})$. By Green's formula,

$$\int_{\partial\omega} v(\zeta)\, d\zeta \overset{\zeta=\xi+i\eta}{=} \int_{\partial\omega} v(\zeta)\, d\xi + iv(\zeta) d\eta = \iint_{\omega} \left( i\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta} \right) L(d\zeta) = 2i \iint_{\omega} \frac{\partial v}{\partial \overline{z}}\, L(d\zeta).$$

Apply this to $v(\zeta) = u(\zeta)/(\zeta - z)$ and $\omega_\varepsilon = \{\zeta \in \omega : |\zeta - z| > \varepsilon\}$ for small $\varepsilon$. We get

$$\int_{\partial\omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \int_{|\zeta - z| = \varepsilon} \frac{u(\zeta)}{\zeta - z}\, d\zeta = 2i \iint_{\omega_\varepsilon} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}(\zeta)\, L(d\zeta).$$

Letting $\varepsilon \to 0^+$, we get

$$\int_{|z-\zeta|=\varepsilon} \frac{u(\zeta)}{\zeta - z}\, d\zeta \to 2\pi i u(z),$$

and

$$\iint_{\omega_\varepsilon} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}\, L(d\zeta) \to \iint_{\omega} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) L(d\zeta) \in L^1$$

by dominated convergence. □